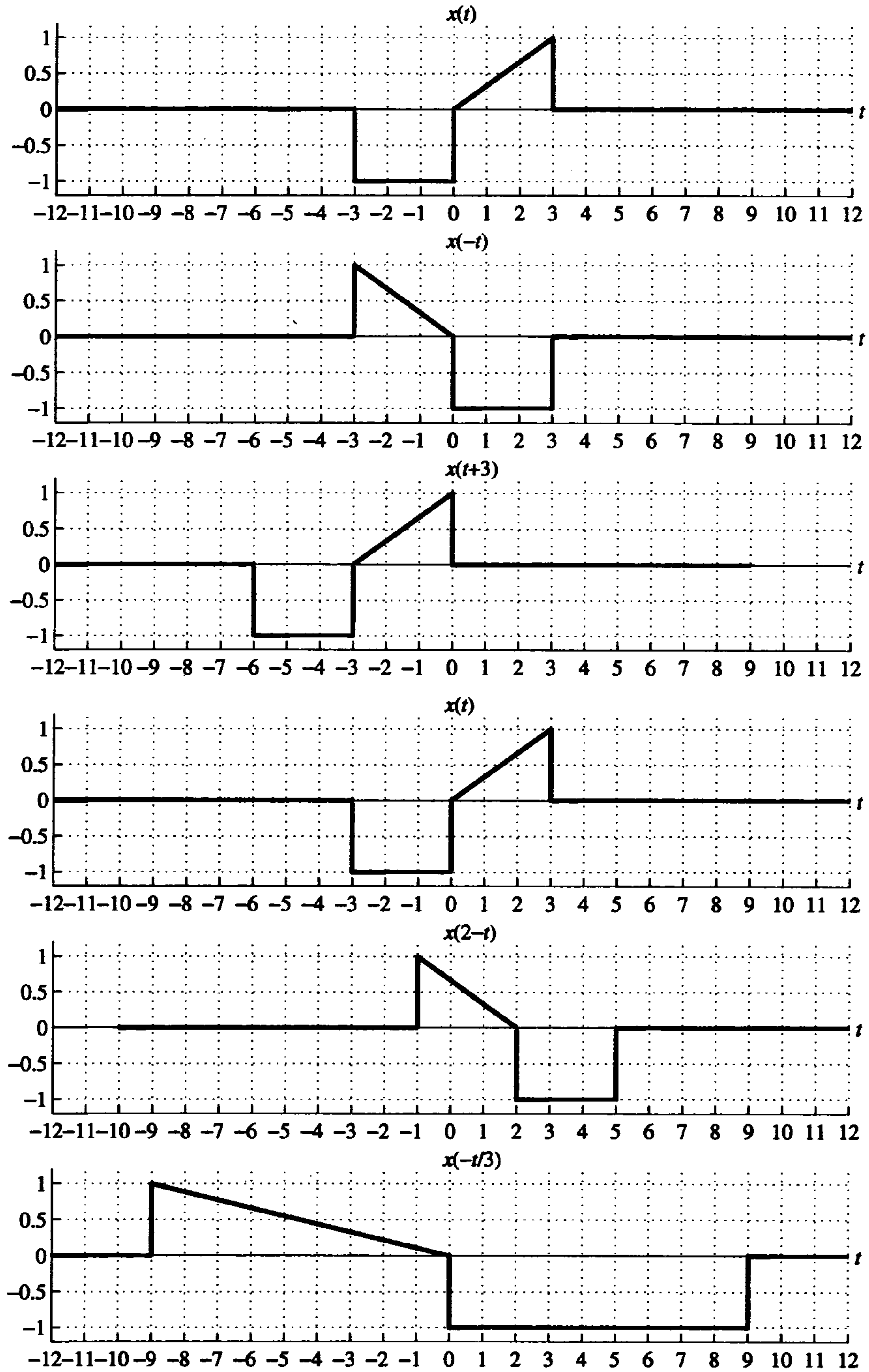
Name: Studen

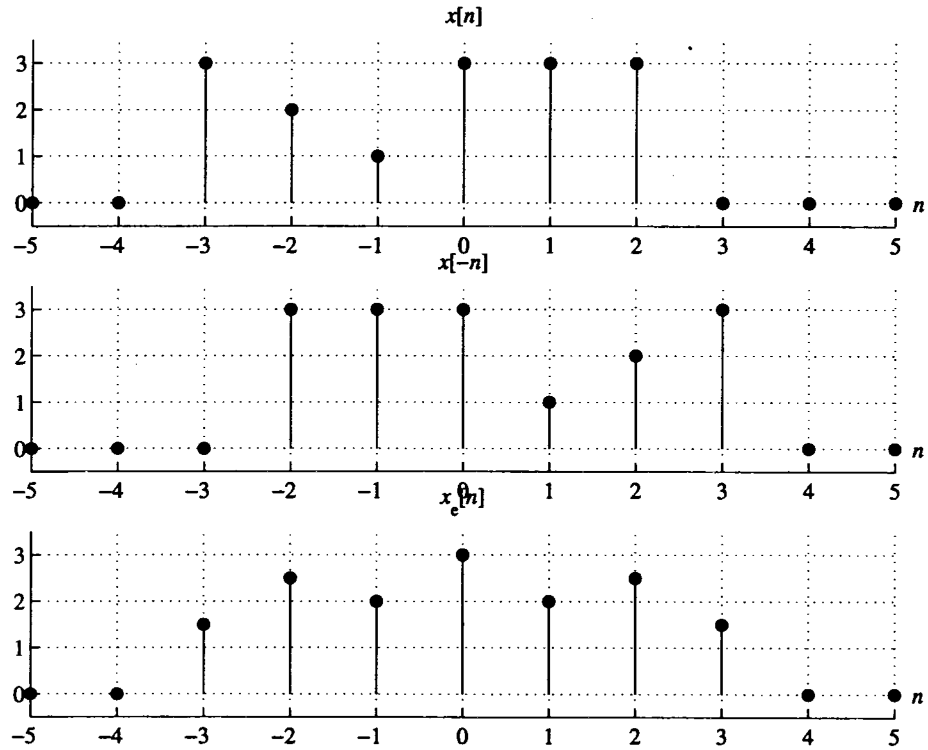
EE.351: Spectrum Analysis and Discr
SOLUTIONS TO MIDTERM EX

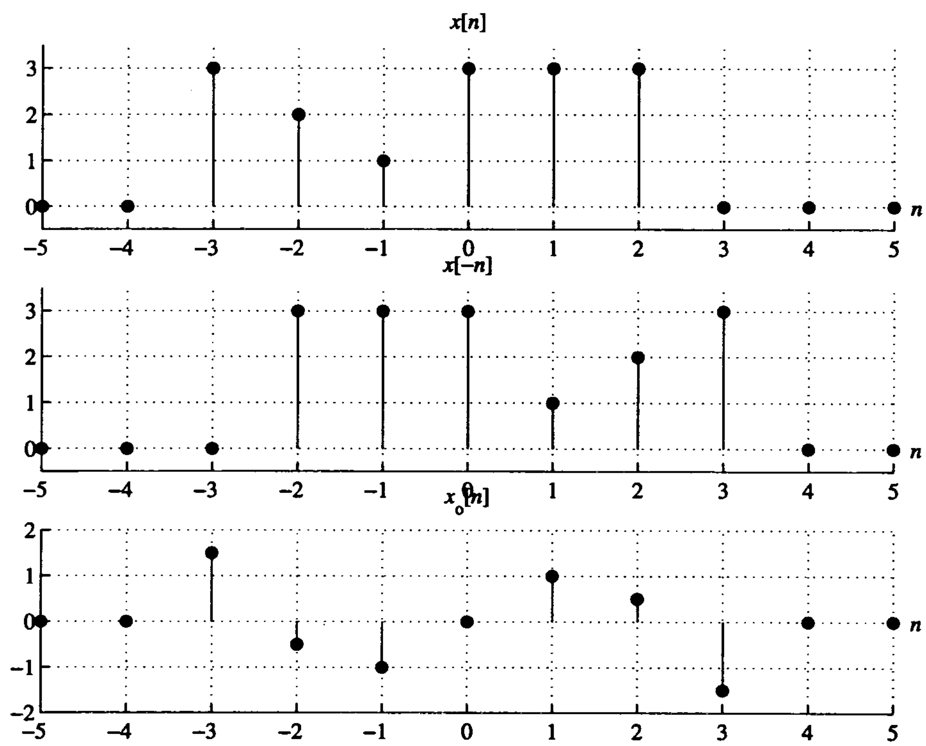
1. *(Signal Transformations)*

[6]



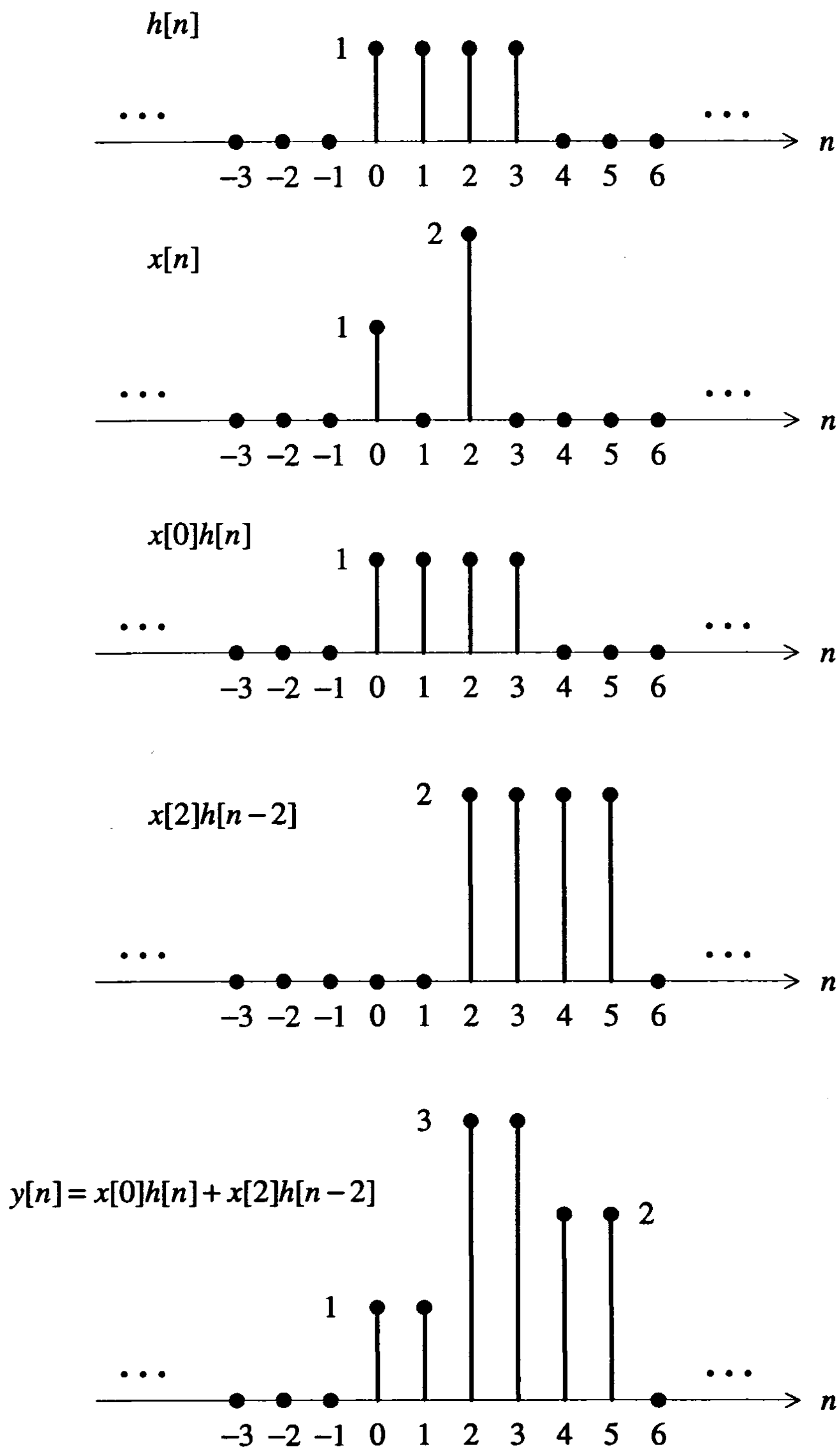
[4]



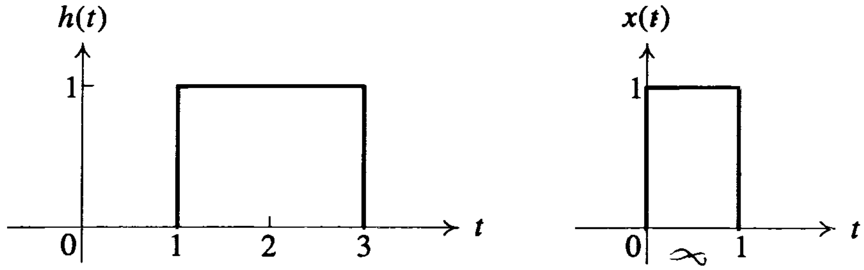


2. *(Convolution)*

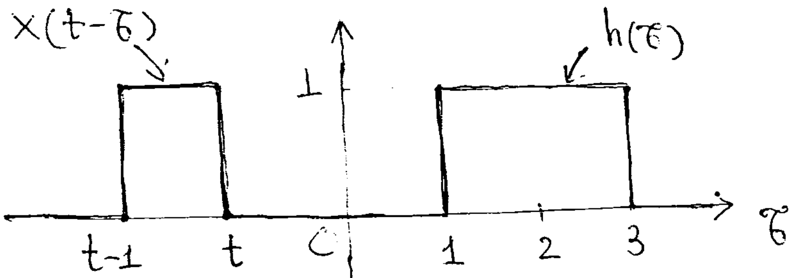
[5] (a) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[0]h[n] + x[2]h[n-2]$.



[5] (b) Consider a continuous-time LTI system with impulse response $h(t)$ and input $x(t)$ as shown below. Find and neatly sketch the output $y(t)$.

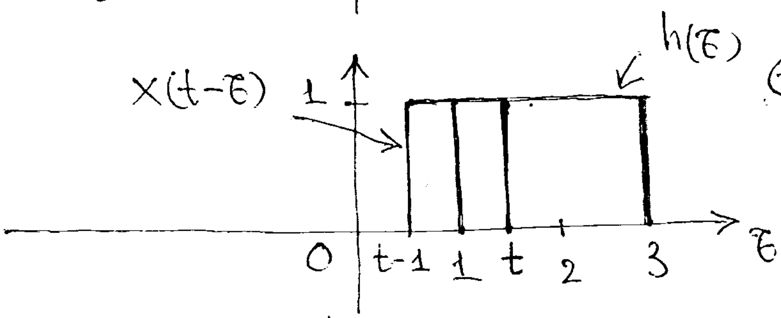


$$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$



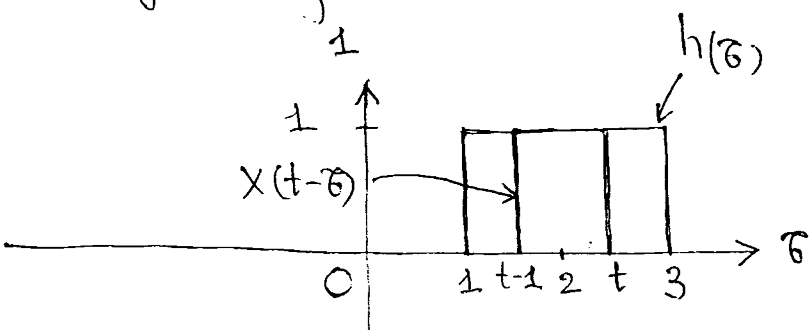
⊕ If $t \le 1$ then

$x(t-\tau)\, h(\tau) = 0 \Rightarrow y(t) = 0$



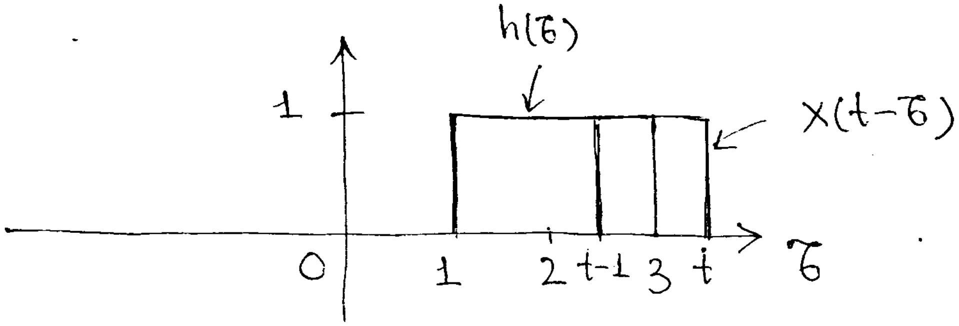
⊕ If $1 \le t \le 2$, then $x(t-\tau)$ and $h(\tau)$ are both non-zero over $1 \le \tau \le t$. Thus

$$y(t) = \int_{1}^{t} h(\tau)\, x(t-\tau)\, d\tau = \int_{1}^{t} d\tau = t-1$$



⊕ If $2 \le t \le 3$, then both $h(\tau)$ and $x(t-\tau)$ are non zero only in the interval $t-1 \le \tau \le t$.

Therefore : $$y(t) = \int_{t-1}^{t} h(\tau)\, x(t-\tau)\, d\tau = \int_{t-1}^{t} d\tau = 1$$

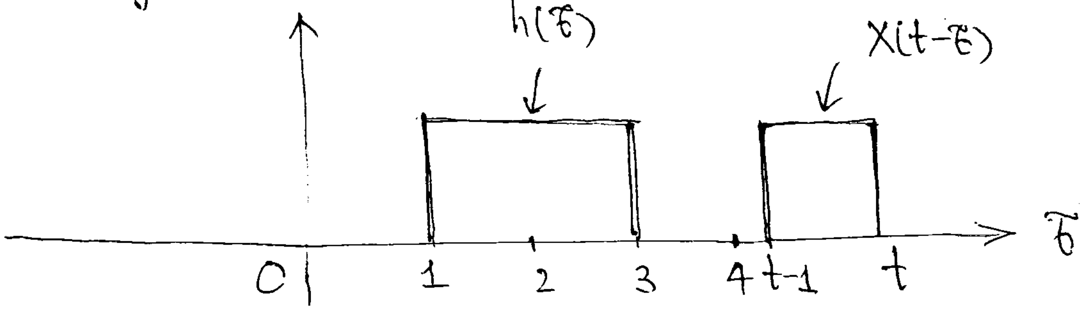


⊕ If $3 \le t \le 4$, then $h(\tau)\,x(t-\tau)$ is nonzero (equal to 1) over $t-1 \le \tau \le 3$. Therefore:

$$y(t) = \int_{t-1}^{3} h(\tau)\,x(t-\tau)\,d\tau = \int_{t-1}^{3} d\tau = 4-t$$

⊕ Finally, if $t > 4$, then $h(\tau)$ and $x(t-\tau)$ are non-overlapped $\Rightarrow$ $h(\tau)\,x(t-\tau) = 0$. Hence

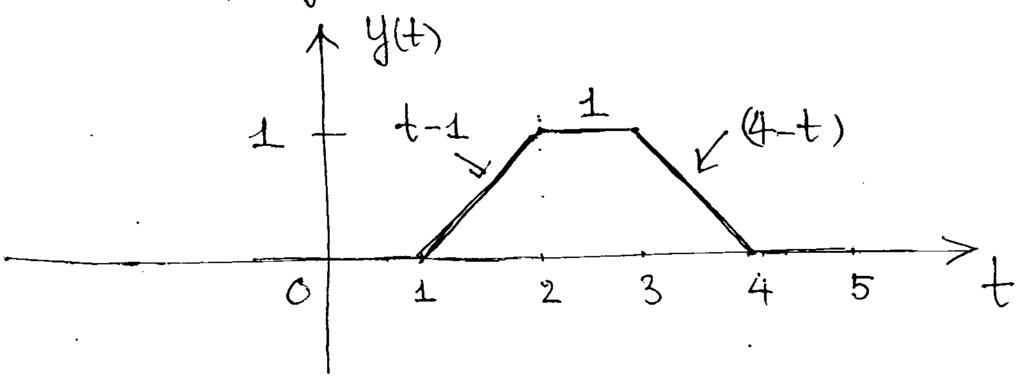
$y(t) = 0$



In summary:

$$y(t) = x(t) * h(t) = \begin{cases} 0 \;; & t \le 1 \text{ and } t \ge 4 \\ t-1 \;; & 1 \le t \le 2 \\ 1 \;; & 2 \le t \le 3 \\ 4-t \;; & 3 \le t \le 4 \end{cases}$$

Sketch of $y(t)$ is shown below:



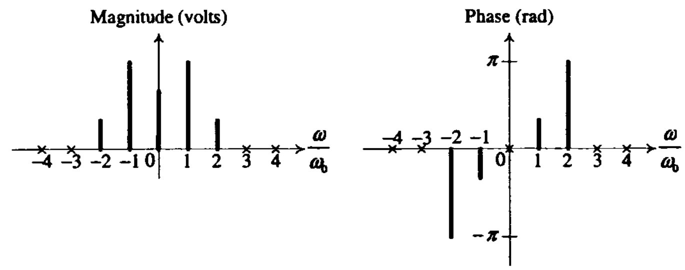
3. *(Properties of Fourier Series Coefficients)*

[8] (a) Signals (i), (iii) and (iv) are real-valued since their magnitude spectra are *even* and phase spectra are *odd.* Signal (ii) is a complex-valued since its phase spectrum is *not* odd.

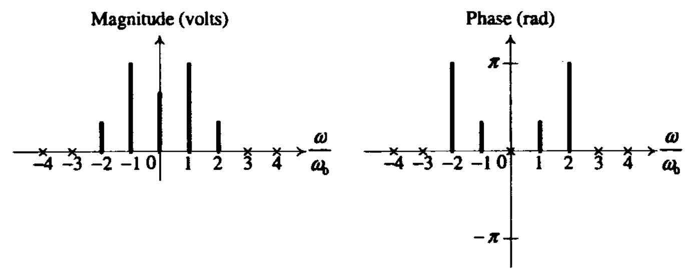
[2] (b) Since $e^{j\pm\pi} = -1$ and $e^{j0} = 1$, the FS coefficients of signal (iii) are real-valued. Thus signal (iii) is both real-valued and even function.

None of the signal is both real-valued and odd since all the four signals have DC component. A real-valued odd signal must have a zero DC component, i.e., $a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\mathrm{d}t = 0$. *Remark:* If the DC component is removed, then the signal (iv) is both real-valued and odd because its FS coefficients are purely imaginary and odd ($e^{\pm j\pi/2} = \pm j$).

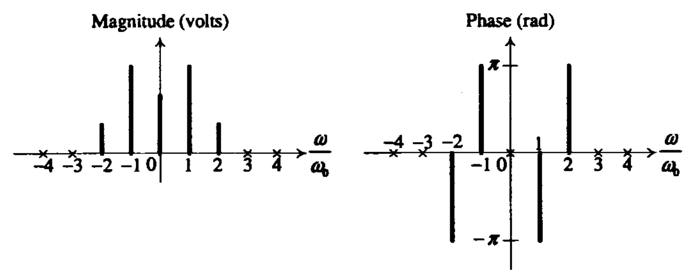
(i)



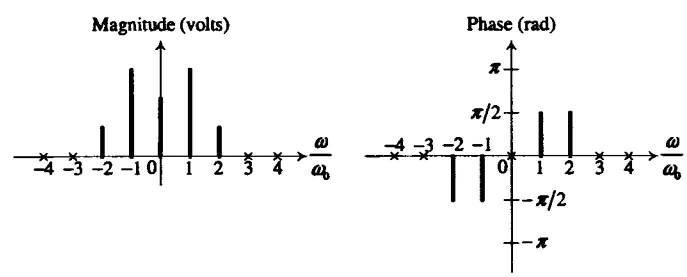
(ii)



(iii)



(iv)



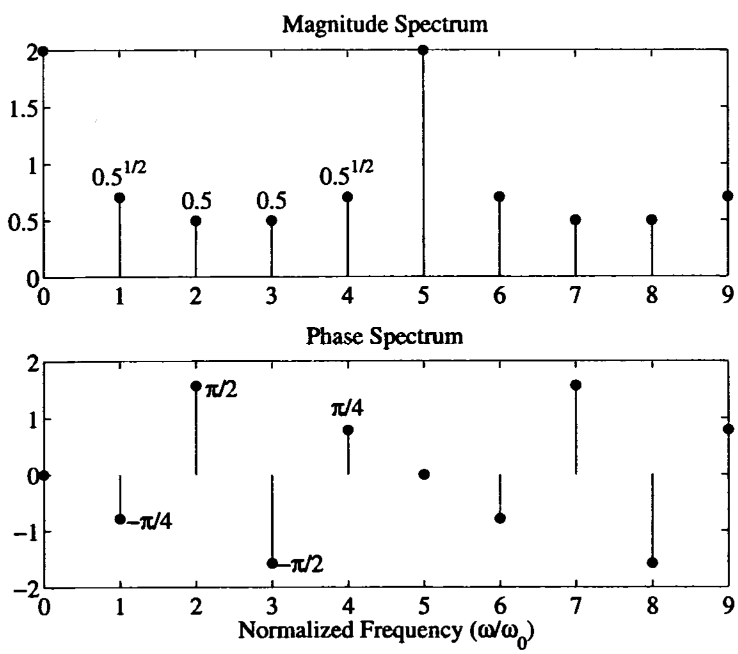
4. *(Fourier Series Representation)*

[5] (a) A discrete-time periodic signal $x[n]$ is real-valued and has a fundamental period $N = 5$. The Fourier series coefficients for $x[n]$ are

$$a_0 = 2, \quad a_1 = a_{-1}^* = \frac{1}{2} - \frac{1}{2}j, \quad a_2 = a_{-2}^* = \frac{1}{2}j$$

(i) Determine and neatly sketch the magnitude and phase spectra of $x[n]$ over at least two periods of a_k.

$$a_0 = 2 \quad \longrightarrow \quad |a_0| = 2, \ \angle a_0 = 0$$
$$a_1 = \frac{1}{2} - \frac{1}{2}j \quad \longrightarrow \quad |a_1| = \frac{\sqrt{2}}{2}, \ \angle a_1 = \arctan(-1) = -\pi/4$$
$$a_2 = \frac{1}{2}j = \frac{1}{2}e^{j\pi/2} \quad \longrightarrow \quad |a_2| = \frac{1}{2}, \ \angle a_2 = \pi/2$$
$$a_3 = a_{3-5} = a_{-2} = -\frac{1}{2}j = \frac{1}{2}e^{-j\pi/2} \quad \longrightarrow \quad |a_3| = \frac{1}{2}, \ \angle a_3 = -\pi/2$$
$$a_4 = a_{4-1} = a_{-1} = \frac{1}{2} + \frac{1}{2}j \quad \longrightarrow \quad |a_4| = \frac{\sqrt{2}}{2}, \ \angle a_4 = \arctan(1) = \pi/4$$

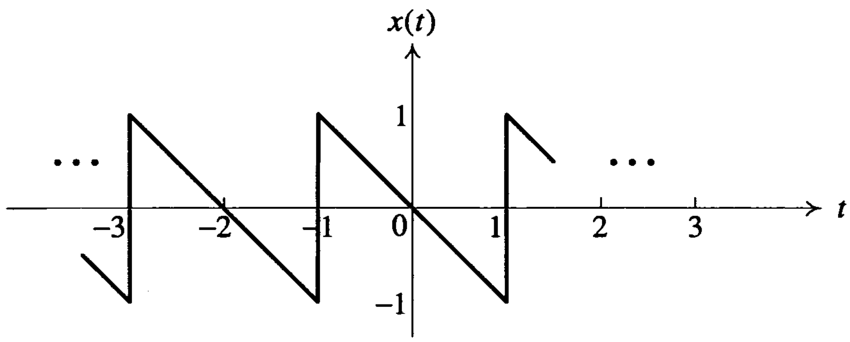


(ii) Express $x[n]$ in the form: $x[n] = A_0 + \sum_{k=1}^{\infty} A_k \cos(\omega_k n + \phi_k)$.

The fundamental frequency is $\omega_0 = 2\pi/5$.

$$\begin{aligned}
x[n] &= \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=-2}^{2} a_k e^{jk\omega_0 n} \\
&= a_0 + [a_{-1}e^{-j\omega_0 n} + a_1 e^{j\omega_0 n}] + [a_{-2}e^{-j2\omega_0 n} + a_2 e^{j2\omega_0 n}] \\
&= 2 + \left[\frac{\sqrt{2}}{2}e^{j\pi/4}e^{-j\omega_0 n} + \frac{\sqrt{2}}{2}e^{-j\pi/4}e^{j\omega_0 n}\right] + \left[\frac{1}{2}e^{-j\pi/2}e^{-j2\omega_0 n} + \frac{1}{2}e^{j\pi/2}e^{j2\omega_0 n}\right] \\
&= 2 + \frac{\sqrt{2}}{2}\left[e^{-j(\omega_0 n - \pi/4)} + e^{j(\omega_0 n - \pi/4)}\right] + \frac{1}{2}\left[e^{-j(2\omega_0 n + \pi/2)} + e^{j(2\omega_0 n + \pi/2)}\right] \\
&= 2 + \sqrt{2}\cos\left(\omega_0 n - \frac{\pi}{4}\right) + \cos\left(2\omega_0 n + \frac{\pi}{2}\right) \\
&= 2 + \sqrt{2}\cos\left(\frac{2\pi}{5}n - \frac{\pi}{4}\right) + \cos\left(\frac{4\pi}{5}n + \frac{\pi}{2}\right)
\end{aligned}$$

[5] (b) Consider the following continuous-time periodic signal $x(t)$



(i) The fund. period is $T_0 = 2$ and the fund. frequency is $\omega_0 = \frac{2\pi}{T_0} = \pi$.

(ii) To compute the Fourier series coefficients a_k of $x(t)$, consider $x(t)$ in one period, from $-1 \le t \le 1$. Then $x(t) = -t, \quad -1 \le t \le 1$. Furthermore, observe that $x(t)$ is a real-valued, odd function. Thus the FS coefficients are purely imaginary. Hence $B_k = 0$ and $a_k = jC_k$, where

$$\begin{aligned}
C_k &= -\frac{1}{T_0}\int_{T_0} x(t)\sin(k\omega_0 t)dt = \frac{1}{2}\int_{-1}^{1} t\sin(\omega_0 kt)dt \\
&= \frac{1}{2}\left[\frac{\sin(\omega_0 kt)}{(\omega_0 k)^2} - \frac{t}{\omega_0 k}\cos(\omega_0 kt)\right]\Bigg|_{-1}^{1} = -\frac{1}{2\pi k}[\cos(k\pi) + \cos(-k\pi)] \\
&= -\frac{\cos(k\pi)}{\pi k} = -\frac{(-1)^k}{\pi k}
\end{aligned}$$

The DC component of $x(t)$ is 0 since $x(t)$ is a real-valued, odd function. To conclude

$$a_k = \begin{cases} 0, & k = 0 \\ -j\dfrac{(-1)^k}{k\pi}, & k = \pm 1, \pm 2, \ldots \end{cases}$$